Bubble Formation in Superposed Magnetic Fluids in the Presence of Heat and Mass Transfer

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In this paper we study the nonlinear Rayleigh-Taylor instability in the case of magnetic fluids in the presence of heat and mass transfer. We find that there is a normal field instability in the linear case. The behaviour of the bubbles in the nonlinear analysis is the same as if they were leaving the surface when the liquid is superheated. The criterion for "Explosive Instability" is also examined.

1. Introduction

Two immiscible fluids, in contact with each other, share a common surface called the interface. Operations involving transfer of matter or of heat across the interface are very common in various physical problems. There are various situations where the effects of heat and mass transfer play an important role, for example, the phenomena of boiling accompanies high heat and mass transfer rates which are significant in determining the flow field and the stability of the system. The Rayleigh-Taylor problem has been a subject of great attention ever since the pioneering work by Lord Rayleigh (see Chandrasekhar [1]). Also the study of magnetic fluids has recieved considerable attention in recent years (see Cowley and Rosensweig [2] and Zelazo and Melcher [3]). The nonlinear analyses attempted so far are mainly concentrated on the study of two semi-infinite fluids separated by the interface under ideal conditions, neglecting dissipative effects.

Hsieh [4, 5] presented a formulation to deal with the interfacial stability problem taking into account heat and mass transfer effects. In the linear analysis, for Rayleigh-Taylor stability problems of a liquid-vapour system, it is found that the effect of heat and mass transfer tends to enhance the stability of the system. Hsieh [6] studied the nonlinear Rayleigh-Taylor stability of a liquid layer over a finite vapour layer. It is observed that the nonlinear effects can indeed increase the range of the stability when the effects of heat and mass transfer are strong enough. The size of bubbles detached from the interface is also estimated for mod-

erately large α, where α is proportional to the heat flux in the system. In this presentation we examine the effect of a normal magnetic field on the nonlinear Rayleigh-Taylor stability problem in magnetic fluids in the presence of heat and mass transfer.

2. Formulation of the Problem

We consider the two-dimensional wave propagation at the interface $y=0$, which separates the two inviscid, incompressible superposed magnetic fluids with the densities $\varrho^{(1)}$ and $\varrho^{(2)}$. The fluids are confined between two parallel planes $y=-h_1$ and $y=h_2$. The interface is taken to be $s(x,t)=y-\eta(x,t)$, where $\eta(x,t)$ is the elevation of the free surface from the unperturbed level. The fluids with magnetic permeabilities μ_1 and μ_2 occupy the regions $y<0$ and $y>0$, respectively. Let the temperatures at $y=-h_1$, $y=0$, and $y=h_2$ be T_1, T_0, and T_2, respectively. The steady magnetic field H_0 acts in the direction normal to the interface. The flow fields is assumed to be irrotational. The velocity potential $\Phi^{(j)}(x,t)$ and magnetic potential $\Psi^{(j)}(x,t)$; $j=1, 2$ in the appropriate region, satisfy Laplace's equation. The vanishing of the normal component of the fluid velocity and the magnetic field at the upper and lower surfaces requires

$$\frac{\partial \phi}{\partial y} = 0 \text{ at } y = -h_1 \text{ and } y = h_2, \tag{1}$$

where ϕ represents Φ or Ψ. The interfacial conditions which express the conservation of mass and momentum are, respectively

$$\Delta[\varrho^{(i)}(\eta_t + \Phi_x^{(i)}\eta_x - \Phi_y^{(i)})] = 0, \tag{2}$$

and

$$\Delta \left[\varrho^{(i)} \left[\Phi_t^{(i)} + \frac{1}{2}((\Phi_x^{(i)})^2 + (\Phi_y^{(i)})^2) + g\eta - \frac{1}{1 + \eta_x^2} \right. \right.$$
$$\left. \left. \cdot (\Phi_x^{(i)} \eta_x - \Phi_y^{(i)})(\eta_t + \Phi_x^{(i)} \eta_x - \Phi_y^{(i)}) \right] \right] \qquad (3)$$
$$= \sigma \eta_{xx} [1 + \eta_x^2]^{-3/2} + \frac{\mu_2 (\mu - 1)}{8\pi} [(\chi_\tau^{(1)})^2 + \mu(\chi_n^{(1)})^2],$$

where $\Delta(\psi^{(i)}) = \psi^{(1)} - \psi^{(2)}$ indicates the jump in ψ across the interface $y = 0$ and $\boldsymbol{\chi}^{(j)} = \boldsymbol{H}_0^{(j)} + \boldsymbol{H}^{(j)}$ denotes the total magnetic field in the region j, H_0 being the steady magnetic field and $H^{(j)}$ is the perturbation; $\mu = \mu_1/\mu_2$.

The normal and tangential components of the magnetic field are continuous; that is

$$\mu_1 \chi_n^{(1)} = \mu_2 \chi_n^{(2)}, \qquad (4)$$

and

$$\chi_\tau^{(1)} = \chi_\tau^{(2)}, \qquad (5)$$

where $\chi_n^{(j)}$ and $\chi_\tau^{(j)}$ denote the normal and tangential components of $\boldsymbol{\chi}^{(j)}$, respectively. The interfacial condition for energy transfer is given by [see 4]

$$\varrho^{(1)}[\eta_t + \Phi_x^{(1)} \eta_x - \Phi_y^{(1)}] = -\alpha(\eta + \alpha_2 \eta^2 + \alpha_3 \eta^3). \qquad (6)$$

The coefficients α_2 and α are given by

$$\alpha_2 = -\left(\frac{1}{h_1} - \frac{1}{h_2}\right) \text{ and } \alpha = \frac{G}{L}\left(\frac{1}{h_1} + \frac{1}{h_2}\right), \quad (7)$$

where G is the equilibrium heat flux and L the latent heat released when the fluid is transformed from one phase to another.

Equations (1)–(7) are the governing equations of the problem. The perturbed quantities have been normalized with respect to the characteristic length $D = [\sigma/g \varrho^{(1)}]^{1/2}$ and characteristic velocity $V = [\sigma g/\varrho^{(1)}]^{1/4}$.

3. Linear Theory

When the interface is perturbed from the equilibrium $y = 0$ to $y = A(t_1, t_2) \exp i(kx - \omega t)$, where A is the amplitude, the dispersion relation for the linearized problem is obtained as

$$\omega^2 [c_{11} + \varrho c_{12}] + i \omega \alpha [c_{11} + c_{12}]$$
$$+ (\varrho - 1)k - k^3 + k^2 \beta_1 = 0, \qquad (8)$$

where use has been made of the boundary condition (4) which leads to the following condition at equilibrium:

$$\mu_1 H_0^{(1)} = \mu_2 H_0^{(2)}, \qquad (8\,a)$$

or

$$H_0^{(1)} = \frac{\mu_2}{\mu_1} H_0^{(2)} = \frac{1}{\mu} H, \qquad (8\,b)$$

where H is the uniform magnetic field and the other symbols mentioned above are given by

$$\varrho = \frac{\varrho^{(2)}}{\varrho^{(1)}}, \ \alpha = \frac{\alpha D}{\varrho^{(1)} V}, \ \beta = \frac{\mu_2 H^2 (\mu - 1)^2 T^{1/2}}{4\pi \mu (\varrho^{(1)} g)^{3/2}},$$
$$c_{ij} = \coth(i k h_j); \ i, j = 1, 2, \qquad (9)$$

and

$$\beta = \beta_1(c_{11} + \mu c_{12}). \qquad (10)$$

We write this equation in the form

$$\omega^2 + 2 i \omega a + b = 0, \qquad (11)$$

where

$$a = \left(\frac{\alpha}{2}\right)\left(\frac{c_{11} + c_{12}}{c_{11} + \varrho c_{12}}\right), \qquad (12)$$

and

$$b = \frac{(\varrho - 1)k - k^3 - k^2 \beta_1}{c_{11} + \varrho c_{12}}. \qquad (13)$$

Equation (11) has the roots $\omega = -ia + (-a^2 - b)^{1/2}$. If $b > 0$, one of the roots of (11) is purely imaginary. This gives rise to instability of the system. If $b < 0$, the system is stable but there is no permanent periodic wave state. If $a > 0$, the growth rate of instability is reduced from that of the case when $a = 0$. The graph between μ and k_c for different sets of depths (h_1, h_2) of fluids are shown in Fig. 1. From the figure it is clear that the system is stable for $\mu_1 < \mu < \mu_2$ when $k > k_c$ and unstable otherwise. The range $\mu_1 < \mu < \mu_2$ decreases with increase in the depths of the fluids. It is clear from the curves in Fig. 1 that if we keep h_1 fixed at a certain value (say 5.8) and increase h_2 from 0.1 to 1.0, the range of μ decreases from $0.5 < \mu < 1.8$ to $0.7 < \mu < 1.5$. Similarly, if we keep h_2 to be say 4.9 and increase h_1 from 0.1 to 1.0, the range decreases from $0.4 < \mu < 1.7$ to $0.7 < \mu < 1.4$. The range $0.7 < \mu < 1.5$ remains fixed if we keep h_1 fixed and increase h_2. Also the range $0.7 < \mu < 1.4$ remains fixed if we keep h_2 fixed and increase h_1. We thus conclude that in the presence of a normal magnetic field, the region of instability increases. In other words, there is a normal field instability.
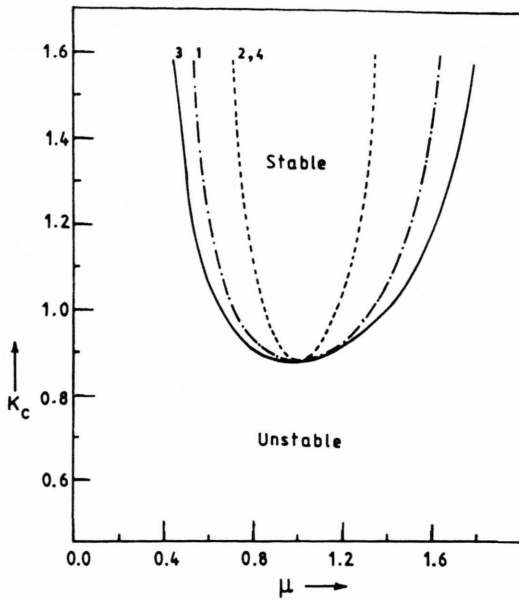
Fig. 1. The behaviour of k_c as a function of μ for different sets of depths of fluids. The curves 1 and 2 represent the graph between μ and k_c for $h_1 = 5.8$ and $h_2 = 0.1$ and $h_1 = 5.8$ and $h_2 = 1.0$, respectively, while the curves 3 and 4 represent the graph between μ and k_c for $h_1 = 0.1$ and $h_2 = 4.9$ and $h_1 = 1.0$ and $h_2 = 4.9$, respectively.

4. Non-Linear Analysis

We shall now study the non-linear behaviour of the system near the critical wavenumber k_c, using the method of multiple scales. We expand the various variables in ascending powers of a small dimensionless parameter ε characterizing the steepness ratio of the wave near the critical wavenumber, letting $k = k_c + \varepsilon^2 \delta$, where $\delta = o(1)$ is a determining parameter. The time variable t is scaled as $t_n = \varepsilon^n t$ and the various physical quantities may be expanded as follows:

$$F(x, y, t) = \sum_{n=1}^{3} \varepsilon^n F_n(x, y, t_0, t_1, t_2) + o(\varepsilon^4). \quad (14)$$

The corresponding critical frequency is zero in this case. We assume that α, α_2, and α_3 are all of order $o(1)$.

Substituting (14) in (1)–(7) and equating the coefficients of equal powers of $\varepsilon^n (n = 1, 2, 3)$, we obtain the linear as well as successive higher order equations, each of which can be solved with the knowledge of solutions of previous orders. These sets of equations are given in Appendix A.

The solutions of the first order problem are

$$\eta_1 = A(t_1, t_2) e^{ikx} + \text{c.c.}, \quad (15)$$

$$\Phi_1^{(j)} = [(-1)^{j-1} \alpha D_{jj} R_{1j}][A e^{ikx} + \text{c.c.}], \quad (16)$$

$$\Psi_1^{(j)} = [(-1)^{j-1}(1-\mu)\mu^{j-2} H S_{1i} E_{11} R_{1j}] \cdot [A e^{ikx} + \text{c.c.}], \quad (17)$$

$$i, j = 1, 2; \quad i \neq j; \quad y_j = y + (-1)^{j-1} h_j.$$

where R_{ij}, S_{1i}, D_{jj} and E_{mm} are as defined in Appendix B (cf. B(1), B(2)). We recover the critical dispersion relation $k = k_c$ to this order (cf: A(6) in Appendix A). Substituting the first order solutions into the second order problem, we obtain the following solvability condition:

$$\frac{\partial A}{\partial t_1} = 0. \quad (18)$$

We thus find the uniformly valid solutions for the second order problem to be:

$$\eta_2 = -2\alpha_2 |A|^2 + k N(A^2 e^{2ikx} + \text{c.c.}), \quad (19)$$

$$\Phi_2^{(i)} = b^{(i)}(t_0, t_1, t_2) + [B_2^{(i)} e^{2ikx} + \text{c.c.}][R_{2j}], \quad (20)$$

$$\Psi_2^{(j)} = W_j R_{2j}[A^2 e^{2ikx} + \text{c.c.}], \quad (21)$$

$$j = 1, 2; \quad y_j = y + (-1)^{j-1} h_j,$$

where the various symbols are given in Appendix B (cf. B(3)–B(7)). Comparing the constant terms in the second order equations, we get the following relation between $b^{(1)}$ and $b^{(2)}$:

$$\begin{aligned}
\frac{\partial b^{(1)}}{\partial t_0} - \varrho \frac{\partial b^{(2)}}{\partial t_0} &= 2(1-\varrho)\alpha_2 |A|^2 \\
&+ \alpha^2 [\varrho^{-1} - [c_{12}^2 - 1] - c_{11}^2 + 1]|A|^2 \\
&- \frac{\beta}{\mu} K^2 [(1-\mu)\mu T_{11}^2 + 1 \\
&+ (1-\mu)c_{11}^2 T_{11}^2 - 2c_{11} T_{11}]|A|^2. \quad (22)
\end{aligned}$$

We now substitute the first and second order solutions into third order equations (A(12)–A(16)); to avoid the non-uniformity of expansions, we obtain a secularity condition in the form

$$\frac{\alpha}{k}[c_{11} + c_{12}]\frac{\partial A}{\partial t_2} + (2k_c \delta + \beta_1 \delta)A + \left(v - \frac{3}{2}k_c^4\right)|A|^2 A = 0, \quad (23)$$

where the quantity v is given in the Appendix B (cf. B(8)).

5. Discussion

Since the phase associated with A remains constant, we can treat A, without loss of generality, as a real quantity. So we obtain from (23)

$$\frac{\mathrm{d}A}{\mathrm{d}t_2} + (a_1 + a_2 A^2) A = 0, \tag{24}$$

where

$$a_1 = \frac{\delta k (2 k_c + \beta_1)}{\alpha (c_{11} + c_{12})}, \tag{25}$$

and

$$a_2 = \frac{k (v - \frac{3}{2} k_c^4)}{\alpha (c_{11} + c_{12})}. \tag{26}$$

Denoting $A(0)$ by A_0, we get

$$A^2(t_2) = a_1 A_0^2 e^{-2a_1 t}[a_1 + a_2 A_0^2 - a_2 A_0^2 e^{-2a_1 t}]^{-1}. \tag{27}$$

With a finite value of A_0, we consider the values of a_1 and a_2 so that the denominator in (27) does not vanish; for these values of the parameter, A is asymptotically bounded.

From (27), we find that the sufficient criterium for stability is $a_2 > 0$, which is due to finite amplitude effects. Stability can also be established if $a_1 > 0$ and

the initial amplitude is small enough, which is the linear result. If $a_2 < 0$; letting $\xi = -a_2$, we have by (24)

$$\frac{\mathrm{d}A}{\mathrm{d}t_2} + (a_1 - \xi A^2) A = 0, \tag{28}$$

which has the asymptotic solution $A_c = (a_1/\xi)^{1/2}$. This expression tells us that when $a_2 < 0$, the amplitude first increases and then becomes asymptotically saturated. We find the solutions of the differential equation (24) with the initial conditions $A(0) = 0.01$ (as an example). These solutions are shown in Figure 2. From the curves of this figure it is clear that (24) has unbounded solutions, in other words "Explosive Instability" occurs in the system.

The criterium for stability is found to be $a_2 > 0$, where

$$a_2 = \frac{k (v - \frac{3}{2} k_c^4)}{\alpha (c_{11} + c_{12})}. \tag{29}$$

When the system is linearly unstable, i.e. when $\delta < 0$, the asymptotic value of A for large times will be given by

$$|A|^2 = \frac{\delta (2 k_c + \beta_1)}{(v - \frac{3}{2} k_c^4)}. \tag{30}$$

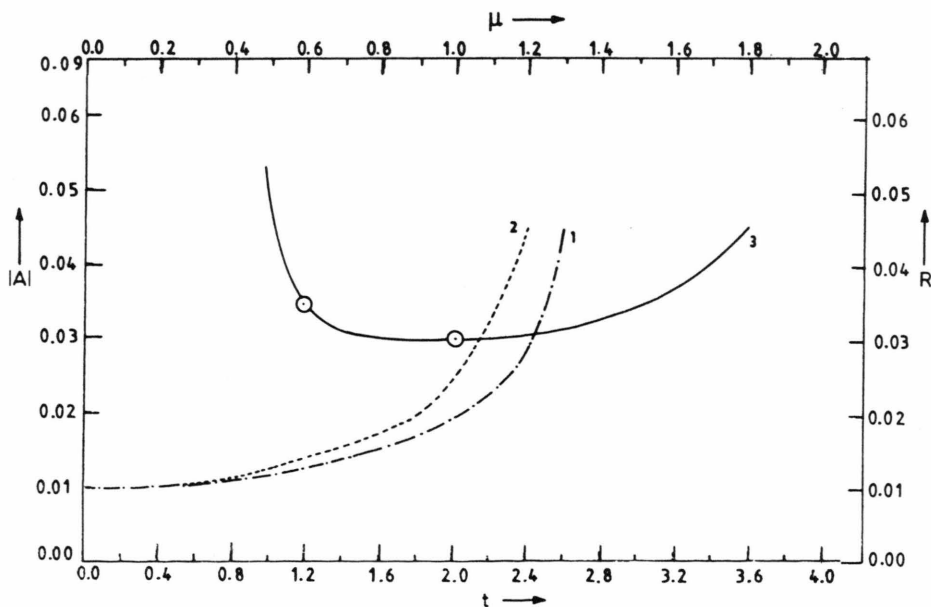Therefore, the range of spectra of the stable wavelength is enlarged by the finite amplitude effect. When

Fig. 2. The curve 3 shows the behaviour of R as a function of μ for $h_1 = 5.8$ and $h_2 = 0.1$ while curves 1 and 2 show the behaviour of $|A|$ as a function of time t for $h_1 = 5.8 \ h_2 = 0.1$ and $\mu = 0.5$, and $h_1 = 5.8 \ h_2 = 0.1$ and $\mu = 1.5$ respectively.

the amplitude exceeds half the wavelength or thickness of the fluid layers, then there is a tendency for bubbles to form and detach from the interface, or to cause the rupture of fluid layers. When the fluid layers are sufficiently thick, the radius R of the bubbles which will detach from the interface is given by

$$R^2 = \left[\delta(2k_c + \beta_1)\left(k_c - \frac{\pi}{R}\right)\right]\left[\left(v - \frac{3}{2}k_c^4\right)^{-1}\right].$$

(31)

In Fig. 2, we see how the radius R of the bubbles varies with the change in the ratio of the magnetic permeabilities μ. It is clear from the figure that R first decreases with increase in μ. The decrease is due to the strong intermolecular forces present inside the fluid. After this decrease in the radius, as shown in the figure there is a bifurcation point at $\mu = 0.59$. This indicates that the bubble leaves the region of strong intermolecular forces, and after leaving this region its radius becomes constant. Again at $\mu = 1.00$ there is a bifurcation point which indicates that the bubble now detachs from the surface, and as soon as it leaves the surface its radius starts increasing with the increase in

μ and attains its final value at $\mu = 1.80$. This behaviour of the bubble, that is increase in the radius of the bubble with increase in μ, coincides with the behaviour of the bubble leaving the surface when the liquid is superheated (see Sissom and Pitts [8], page 639–640).

If h_1 is smaller than the thickness of the fluid layers, then the criterium against rupture of the fluid layers is giving by

$$h_1^2 > \left|\frac{\delta(2k_c + \beta_1)}{(v - \frac{3}{2}k_c^4)}\right|.$$

(32)

For semi-infinite fluid layers it may be seen that v vanishes. Thus the system cannot be stabilized by finite amplitude effects. This is due to the fact that when thickness of the layer becomes very large, the effects of heat and mass transfer become negligible.

Acknowledgement

Appendix A

Order ε^n; $n = 1, 2, 3$,

$$\nabla^2 \Phi_n^{(j)} = 0, \ \nabla^2 \Psi_n^{(j)} = 0; j = 1, 2,$$

A(1)

$$\Phi_{n_y}^{(j)} = 0, \ \Psi_{n_y}^{(j)} = 0 \text{ on } y = -h_1 \text{ and } y = h_2; j = 1, 2,$$

A(2)

where

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}.$$

A(3)

The interfacial conditions on $y = 0$ are given by

Order ε

$$\Delta[\varrho^{(i)} \mathit{L}(\eta_1, \Phi_1^{(i)})] = 0, \ \varrho^{(1)} \mathit{L}(\eta_1, \Phi_1^{(1)}) = -\alpha \eta_1,$$

A(4)

$$\mu \Psi_{1_y}^{(1)} - \Psi_{1_y}^{(2)} = 0, \ [\Psi_{1_x}^{(1)} - \Psi_{1_x}^{(2)}] = \eta_{1_x} H\left(\frac{1 - \mu}{\mu}\right),$$

A(5)

$$\Delta[\varrho^{(i)} D(\eta_1, \Phi_1^{(i)})] = \sigma \eta_{1_{xx}} - 2H\left(\frac{\mu_2(\mu - 1)}{8\Pi}\right)\Psi_{1_y}^{(1)}.$$

A(6)

Order ε^2

$$\Delta[\varrho^{(i)}[\mathit{L}(\eta_2, \Phi_2^{(i)}) + M(\eta_1, \Phi_1^{(i)})]] = 0,$$

A(7)

$$\mu \Psi_{2_y}^{(1)} - \Psi_{2_y}^{(2)} = -\eta_1[\mu \Psi_{1_{yy}}^{(1)} - \Psi_{1_{yy}}^{(2)}] + n_{1_x}[\mu \Psi_{1_x}^{(1)} - \Psi_{1_x}^{(2)}],$$

A(8)

$$\Psi_{2_x}^{(1)} - \Psi_{2_x}^{(2)} = \eta_{2_x} H\left(\frac{1 - \mu}{\mu}\right) + \eta_1[\Psi_{1_{xy}}^{(2)} - \Psi_{1_{xy}}^{(1)}] - \eta_{1_x}[\Psi_{1_y}^{(1)} - \Psi_{1_y}^{(2)}],$$

A(9)

$$\varrho^{(1)}[\pounds(\eta_2, \Phi_2^{(1)}) + M(\eta_1, \Phi_1^{(1)})] = -\alpha(\eta_2 + \alpha_2 \eta_1^2), \tag{A(10)}$$

$$\Delta[\varrho^{(i)}[D(\eta_2, \Phi_2^{(i)}) + E(\eta_1, \Phi_1^{(i)})]]$$
$$= \sigma \eta_{2_{xx}} + \left(\frac{\mu_2(\mu-1)}{8\pi}\right)[\mu(\Psi_{1_y}^{(1)})^2 + 2H\eta_{1_x}\Psi_{1_x}^{(1)} - 2H(\Psi_{2_y}^{(1)} + \eta_1\Psi_{1_{yy}}^{(1)})$$
$$-\left(\frac{H}{\mu}\right)^2(\mu-1)(\eta_{1_x})^2 + (\Psi_{1_x}^{(1)})^2 - 2\frac{H}{\mu}\eta_{1_x}\Psi_{1_x}^{(1)}]. \tag{A(11)}$$

Order ε^3

$$\Delta[\varrho^{(i)}[\pounds(\eta_3, \Phi_3^{(i)}) + N(\eta_1, \Phi_1^{(i)})]] = 0, \tag{A(12)}$$

$$\mu\Psi_{3_y}^{(1)} - \Psi_{3_y}^{(2)} = -\eta_1[\mu\Psi_{2_{yy}}^{(1)} - \Psi_{2_{yy}}^{(2)}] - \eta_2[\mu\Psi_{1_{yy}}^{(1)} - \Psi_{1_{yy}}^{(2)}] - \frac{1}{2}\eta_1^2[\mu\Psi_{1_{yyy}}^{(1)} - \Psi_{1_{yyy}}^{(2)}]$$
$$+ \eta_{1_x}[\mu\Psi_{2_x}^{(1)} - \Psi_{2_x}^{(2)}] + \eta_{2_x}[\mu\Psi_{1_x}^{(1)} - \Psi_{1_x}^{(2)}] + \eta_1\eta_{1_x}[\mu\Psi_{1_{xy}}^{(1)} - \Psi_{1_{xy}}^{(2)}], \tag{A(13)}$$

$$\Psi_{3_x}^{(1)} - \Psi_{3_x}^{(2)} = \eta_2[\Psi_{1_{xy}}^{(2)} - \Psi_{1_{xy}}^{(1)}] + \eta_1[\Psi_{2_{xy}}^{(2)} - \Psi_{2_{xy}}^{(1)}]$$
$$+ \frac{1}{2}\eta_1^2[\Psi_{1_{xyy}}^{(2)} - \Psi_{1_{xyy}}^{(1)}] + H\left(\frac{1-\mu}{\mu}\right)\eta_{3_x} - \eta_{1_x}[\Psi_{2_y}^{(1)} - \Psi_{2_y}^{(2)}]$$
$$- \eta_{2_x}[\Psi_{1_y}^{(1)} - \Psi_{1_y}^{(2)} - \eta_1\eta_{1_x}[\Psi_{1_{yy}}^{(1)} - \Psi_{1_{yy}}^{(2)}], \tag{A(14)}$$

$$\varrho^{(1)}[\pounds(\eta_3, \Phi_3^{(1)}) + N(\eta_1, \Phi_1^{(1)})] = -\alpha(\eta_3 + 2\alpha_2\eta_1\eta_2 + \alpha_3\eta_1^3), \tag{A(15)}$$

$$\Delta[\varrho^{(i)}[D(\eta_3, \Phi_3^{(i)}) + G(\eta_1, \Phi_1^{(i)}) + H(\eta_1, \Phi_1^{(i)}) - I(\eta_1, \Phi_1^{(i)})]]$$
$$= \sigma\left[\eta_{3_{xx}} - \frac{3}{2}\eta_{1_{xx}}\eta_{1_x}^2\right] + \left(\frac{\mu_2(\mu-1)}{8\pi}\right)\left[-\frac{2}{\mu}H^2\eta_{1_x}\eta_{2_x} + 2\mu\Psi_{1_y}^{(1)}(\Psi_{2_y}^{(1)} + \eta_1\Psi_{1_{yy}}^{(1)})\right.$$
$$+ 2H(\eta_{2_x}\Psi_{1_x}^{(1)} + \eta_{1_x}\Psi_{2_x}^{(1)} + \eta_1\eta_{1_x}\Psi_{1_{xy}}^{(1)}) - 2H(\Psi_{3_y}^{(1)} + \eta_2\Psi_{1_{yy}}^{(1)}$$
$$+ \eta_1\Psi_{2_{yy}}^{(1)} + \frac{1}{2}\eta_1^2\Psi_{1_{yyy}}^{(1)}) - 2\mu\eta_{1_x}\Psi_{1_x}^{(1)}\Psi_{1_y}^{(1)} + 2H(\eta_{1_x})^2\Psi_{1_y}^{(1)} + 2\left(\frac{H}{\mu}\right)^2\eta_{1_x}\eta_{2_x} + 2\Psi_{1_x}^{(1)}\Psi_{2_x}^{(1)}$$
$$+ 2\eta_1\Psi_{1_x}^{(1)}\Psi_{1_{xy}}^{(1)} - 2\frac{H}{\mu}(\eta_{1_x})^2\Psi_{1_y}^{(1)} - 2\frac{H}{\mu}(\eta_{2_x}\Psi_{1_x}^{(1)}$$
$$+ \eta_{1_x}\Psi_{2_x}^{(1)} + \eta_1\eta_{1_x}\Psi_{1_{xy}}^{(1)}) + 2\eta_{1_x}\Psi_{1_x}^{(1)}\Psi_{1_y}^{(1)}], \tag{A(16)}$$

where

$$\pounds(\eta_i, \Phi_i^{(j)}) = \eta_{i_{t_0}} - \Phi_{i_y}^{(j)}, \quad D(\eta_i, \Phi_i^{(j)}) = \Phi_{i_{t_0}}^{(j)} + g\eta_i,$$

$$M(\eta_i, \Phi_i^{(j)}) = \eta_{i_{t_1}} + \Phi_{i_x}^{(j)}\eta_{i_x} - \eta_i\Phi_{i_{yy}}^{(j)},$$

$$E(\eta_i, \Phi_i^{(j)}) = \Phi_{i_{t_1}}^{(j)} + \eta_i\Phi_{i_{t_{0y}}}^{(j)} + \frac{1}{2}(\Phi_{i_y}^{(j)})^2 + \frac{1}{2}(\Phi_{i_x}^{(j)})^2 - \Phi_{i_y}^{(j)}(\Phi_{i_y}^{(j)} - \eta_{i_{t_0}}),$$

$$N(\eta_i, \Phi_i^{(j)}) = \eta_{(i+1)_{t_1}} + \eta_{i_{t_2}} + \Phi_{i_x}^{(j)}\eta_{(i+1)_x} + \eta_{i_x}(\Phi_{(i+1)_x}^{(j)}$$
$$+ \Phi_{i_{xy}}^{(j)}\eta_i) - (\Phi_{i_{yy}}^{(j)}\eta_{(i+1)} + \Phi_{(i+1)_{yy}}^{(j)}\eta_i + \frac{1}{2}\eta_i^2\Phi_{i_{yyy}}^{(j)}),$$

$$G(\eta_i, \Phi_i^{(j)}) = \Phi_{(i+1)_{t_1}}^{(j)} + \Phi_{i_{t_2}}^{(j)} + \Phi_{i_{t_{0y}}}^{(j)}\eta_{(i+1)} + \eta_i(\Phi_{(i+1)_{t_{0y}}}^{(j)} + \Phi_{i_{t_{1y}}}^{(j)}) + \frac{1}{2}\eta_i^2\Phi_{i_{t_{0yy}}}^{(j)},$$

$$H(\eta_i, \Phi_i^{(j)}) = \frac{1}{2}\{2\Phi_{i_x}^{(j)}\Phi_{(i+1)_x}^{(j)} + \{\eta_i(\Phi_{i_x}^{(j)})^2\}_y\} + \frac{1}{2}\{2\Phi_{i_y}^{(j)}\Phi_{(i+1)_y}^{(j)} + \{\eta_i(\Phi_{i_y}^{(j)})^2\}_y\},$$

$$I(\eta_i, \Phi_i^{(j)}) = \pounds(\eta_1, \Phi_1^{(1)})(\Phi_{i_x}^{(j)}\eta_{i_x} - \Phi_{(i+1)_y}^{(j)}) - \Phi_{i_y}^{(j)}[\eta_{(i+1)_{t_0}} + \eta_{i_{t_1}} - \Phi_{(i+1)_y}^{(j)} + \Phi_{i_x}^{(j)}\eta_{i_x}]$$
$$+ n_i\frac{\partial}{\partial y}[\Phi_{i_y}^{(j)}\pounds(\eta_1, \Phi_1^{(1)})]. \tag{A(17)}$$

Appendix B

$$R_{lj} = \cosh(l\,k\,y_j),\; S_{li} = \sinh(l\,k\,h_i),\; D_{jj}^{-1} = \varrho^{(j-1)}\,k\,\sinh(k\,h_j), \tag{B(1)}$$

$$E_{mm}^{-1} = \cosh(m\,k\,h_1)\sinh(m\,k\,h_2) + \mu\sinh(m\,k\,h_1)\cosh(m\,k\,h_2), \tag{B(2)}$$

$$B_2^{(i)} = (-1)^{i-1}\,\tau_i\,A^2\,[k\,N + \alpha_2 + (-1)^i\,2\,k\coth(k\,h_i)], \tag{B(3)}$$

$$W_1 = (1 - \mu)\,H\,k\left[\frac{1}{\mu}N\sinh(2\,k\,h_2)\,E_{22} + \frac{(1-\mu)}{\mu}\sinh(2\,k\,h_2)\,E_{22}\,T_{11}\right.$$
$$\left.- \cosh(2\,k\,h_2)\coth(k\,h_1)\,E_{22}\,T_{11} - \cosh(2\,k\,h_2)\coth(k\,h_2)\,E_{22}\,T_{11}\right], \tag{B(4)}$$

$$W_2 = \gamma_1\,W_1 - \frac{1}{\mu}(1-\mu)\,H\,k\,\gamma_2\,[(1-\mu)\,W_3\,E_{11} + N], \tag{B(5)}$$

$$N = \beta_3\left[\frac{1}{2\,k}\alpha^2\,\odot_1 - k\,\mu\,\beta\left(\frac{1}{2\,\mu}(1 - \mu - 2\coth(k\,h_2))\,T_{11} - \frac{1}{\mu^2}\right.\right.$$
$$-\frac{2}{\mu}(1-\mu)\,T_{11}\,T_{22} + 2\,[\coth(k\,h_1)\coth(2\,k\,h_2) + \coth(2\,k\,h_2)]\,T_{22}\,T_{11}$$
$$\left.\left.-\frac{1}{\mu^2}(1-\mu)\coth(k\,h_1)\,T_{11} - \frac{1}{2\,\mu^2}(1-\mu)\coth^2(k\,h_1)\,T_{11}^2)\right)\right], \tag{B(6)}$$

where

$$\tau_i = \frac{\alpha\,\varrho^{1-i}}{2\,k\sinh(2\,k\,h_i)},\; \gamma_2 = \frac{1}{\cosh(2\,k\,h_2)},\; \gamma_1 = \frac{\cosh(2\,k\,h_1)}{\cosh(2\,k\,h_2)},$$

$$\beta_3 = \frac{1}{3\,(\varrho - 1) + 4\,k\,\beta_1 - 2\,k\,\beta\,T_{22}},\; T_{ii}^{-1} = \coth(i\,k\,h_1) + \mu\coth(i\,k\,h_2),$$

$$\odot_1 = 1 + \coth^2(k\,h_1) - \left(\frac{1 + \coth^2(k\,h_2)}{\varrho}\right),\; W_3 = \sinh(k\,h_1)\sinh(k\,h_2). \tag{B(7)}$$

$$v = \alpha^2\,k\left[\left(N + \frac{\alpha_2}{k}\right)\right][\coth(p_1) - \varrho^{-1}\coth(p_2)] - 2\,[\coth(p_1)\coth(k\,h_1)$$
$$+ \varrho^{-1}\coth(p_2)\coth(k\,h_2)] + \mu\,\beta\,k^3\left[-\frac{2\,N}{\mu(1-\mu)}\right] + 2\,(1-\mu)^2\,T_{11}\,T_{22}$$
$$+ \frac{2}{\mu}(1-\mu)\,N\,T_{11}\,T_{22} - 2\,(1-\mu)\coth(2\,k\,h_2)[\coth(k\,h_1) + \coth(k\,h_2)]\,T_{11}\,T_{22}$$
$$+ \frac{3}{\mu}\coth(k\,h_1)\,T_{11} + \frac{2}{\mu}(N+1)\,T_{11} - 2\,(1-\mu)\coth(2\,k\,h_1)\coth(k\,h_2)\,T_{11}^2\,T_{22}$$
$$- \frac{2}{\mu}(1-\mu)\coth(2\,k\,h_1)\coth(k\,h_2)\,T_{11}\,T_{22}\{N + \mu\coth(2\,k\,h_2)[\coth(k\,h_1)$$
$$+ \coth(k\,h_2)]\,T_{11}\} - \frac{2}{\mu}\coth(k\,h_2)\,T_{11}\{\mu(1-\mu)\,T_{11} + N\} - \coth(k\,h_2)[\coth(k\,h_1)$$
$$+ \coth(k\,h_2)]\,N\,T_{11}^2 + \frac{2}{K}\alpha_2\coth(k\,h_2)[\coth(k\,h_1) + \coth(k\,h_2)]\,T_{11}^2$$
$$- \frac{1-\mu}{\mu}\,T_{11}^2\left\{N + \frac{2}{K}\alpha_2\right\} + 2\,\gamma_3\{\coth(2\,k\,h_1) + \coth(2\,k\,h_2)\}$$

$$\cdot T_{11}^2 \, T_{22} \, [k_2 \, (1 - \mu) + N] - \frac{2}{\mu} (1 - \mu) \gamma_3 \, T_{11}^2$$

$$- \, 2 \coth(2 \, k \, h_2) \, [\coth(k \, h_1) + \coth(k \, h_2)] \, [\coth(2 \, k \, h_1) + \coth(2 \, k \, h_2)] \, \gamma_3 \, T_{11}^2 \, T_{22}$$

$$- \frac{2}{\mu} N \gamma_3 \, T_{11}^2 - \frac{1}{\mu} N \coth(k \, h_1) \, T_{11} - \frac{3}{2 \mu} T_{11} + \frac{2 N}{\mu^2 (1 - \mu)}$$

$$+ \frac{2}{\mu^2} (1 - \mu) \, T_{11} \, T_{22} \, [\mu (1 - \mu) \, T_{11} + N] - \frac{2}{\mu} (1 - \mu) \coth(2 \, k \, h_2) \, [\coth(k \, h_1)$$

$$+ \coth(k \, h_2)] \, T_{11}^2 \, T_{22} - \frac{1}{\mu^2} T_{11} - \frac{2}{\mu^2} N \, [T_{11} + \coth(2 \, k \, h_1) \, T_{22}]$$

$$- \frac{2}{\mu} \coth(2 \, k \, h_1) \, T_{11} \, T_{22} \, [(1 - \mu) - \coth(2 \, k \, h_2) \{\coth(k \, h_1) + \coth(k \, h_2)\}]$$

$$+ \frac{1}{\mu^2} (1 - \mu) \, T_{11} + \frac{1}{2} \frac{(1 + \mu)}{\mu} c_{12} \, T_{11}^2, \qquad \qquad \text{B(8)}$$

where

$$\coth(p_i) = [\coth(k \, h_i) \coth(2 \, k \, h_i) - 1], \quad \gamma_3 = \frac{1}{\coth(2 k \, h_2)} . \qquad \qquad \text{B(9)}$$

[1] S. Chandrasekhar, Hydrodynamic and Hydromagnetic Stability, Oxford University Press, Oxford, England 1961.
[2] M. D. Cowley and R. E. Rosensweig, J. Fluid Mech., **30,** 671 (1967).
[3] R. E. Zelago and J. R. Melcher, J. Fluid Mech., **39,** 1 (1969).
[4] D. Y. Hsieh, Trans. ASME, **94 D,** 156 (1972).
[5] D. Y. Hsieh, Phys. of Fluids, **21,** 745 (1978).
[6] D. Y. Hsieh, Phys. of Fluids, **22**(8), 1435 (1979).
[7] L. E. Sissom and D. R. Pitts, Elements of Transport Phenomena, McGraw Hill Book Company, New York 1972.